Microelectronics System Design Research Group
School of Electrical and Electronic Engineering
Merz Court
Newcastle University
Newcastle upon Tyne, NE1 7RU, UK

http://async.org.uk/

Anecdotes about hyperbolas (…and how they reveal themselves in our lives and in electronic systems)

Alex Yakovlev

alex.yakovlev@ncl.ac.uk

NCL-EEE-MSD-MEMO-2013-007

August 2013

Abstract

This memo talks about an emerging paradigm of computing using hyperbolas that are a result of the dynamic behaviour in systems consisting of a container (capacitor, bank account …) and a discrete event generator whose level of activity is modulated by the amount stored in the container (pressure, charge, amount of money…). It attempts to raise motivation for studying this phenomenon from some basic principles and analogues in real world (economy, physics, biology).

Anecdote 1: Shopping with shrinking bank account and motivation

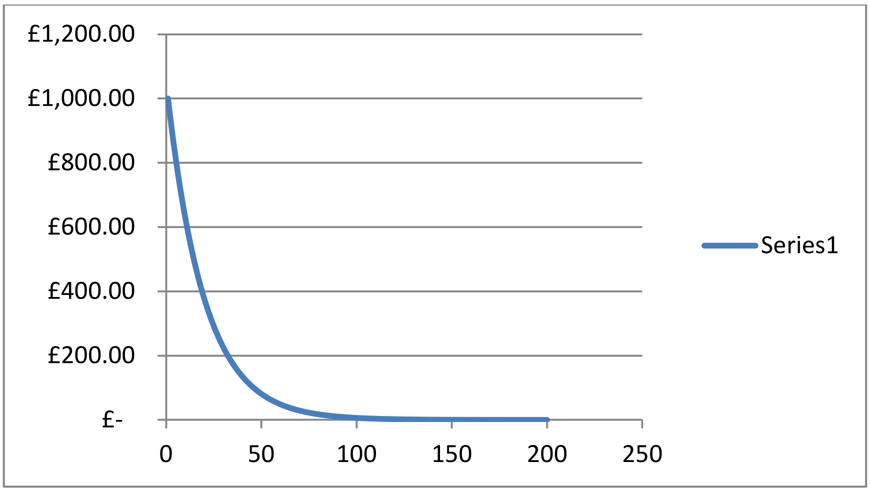
I once approached my wife Maria with the following question:

"Suppose I open a bank account for you to do shopping. I credit it with £1000 pounds. The rule of the shopping exercise is that each shopping act is a discrete event during which you spend exactly 5% of the current value of the account. Namely, your first shopping act spends £50 (5% of £1000), the next one £47.50 (5% of £950), then £45.13 (5% of £902.50). The account will then start to be thinner and thinner. For example, after the 20th shopping act, there will only be £358.47 left. Now, my question is basically, as your account becomes significantly thinner will you shop with the same frequency as in the beginning when you had much money, in other words, will then the intervals between shopping actions be longer or remain the same? "

After a quick deliberation Maria's answered: "I will probably start shopping less frequently as I would want my money to last longer. In fact I would certainly do that if I used not the 5% of the account but a fixed sum on each of my purchases".

This is an interesting answer, which makes you think that the amount of money on the account somehow determines the level of Maria's shopping activity. Presumably, it also reflects some sort of inertia inherent in all of us as we have less incentive to do things when we have less pressure. Indeed, the less money is on the account the slower is our reaction on acting as we somehow subconsciously weigh the worthiness of this smaller scale shopping against the need to overcome the obstacles, such as leaving the house, taking a bus and visiting the shop.

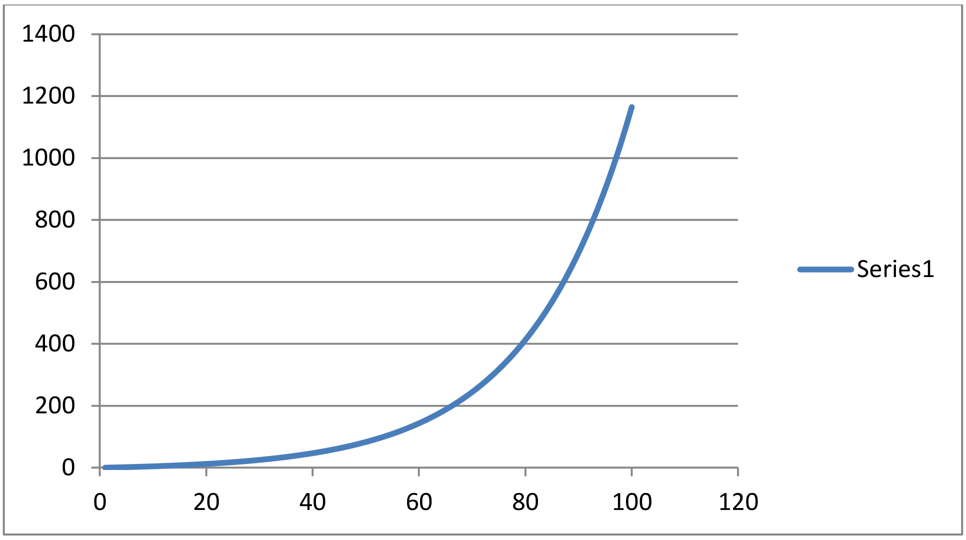
Plotting the current value of the account against shopping events (where 5% of the current value is spent) gives us an exponential shown in the figure below. The exponential is an approximation of (0.95)^N, where N is the shopping act index.



Plot 1. The value of the account as a function of the shopping act index

Remark. *This calculation was performed in Excel, with rounding the monetary value to a penny, hence at some point when the current value becomes 9 pence, the shopping would stop because its 5% (spending share) becomes less than one penny. So, if we strictly follow the rules of the game, at the level of 9p the frequency of shopping drops to zero (of course, if we used real numbers, we will run into difficulty here!)*

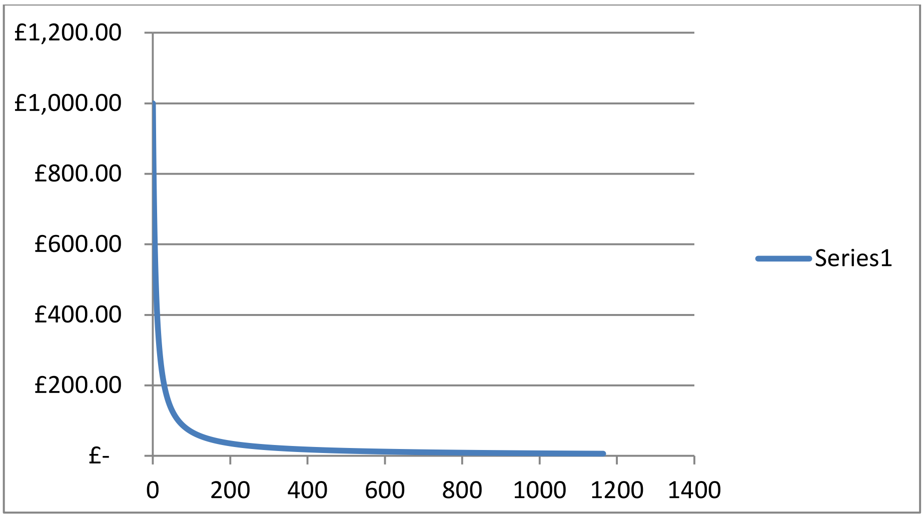
What can be observed that in this graph the x-axis is shopping number but not time. Now, if we follow the line of reasoning that comes from Maria's answer, we might want to see how the account depreciates in time. But for that we need to perform extra work. We need to model the length of time as being the function of the current account value. For simplicity, I decided to model time interval to be inversely proportional to the latest account value. For example, when the value is initially £1000, let's assume that the first shopping act happens on the time whose numerical value is 365*1/1000=0.365 in days, i.e. after 0.365*24=8.76 hours (which is 8hrs:45min:36sec) from the start. If we use the rule of inverse proportionality and apply a year as our basic unit, we will have that between the 20th and 21st shopping acts, there will pass 23hrs:12min:54sec, which is pretty close to one day. Then between the 100th and 101st shopping acts, when the account will only have £6.23, and Maria's buying capacity is limited to 31 pence (one apple), there will be nearly 59 days. Clearly, Maria will have lost much interest in using this account for shopping by then. Further, at the hypothetical fixed point when the account has 9p only, the next shopping event after the previous one will be due in 4055 days, i.e. more than 11 years. So, the exponential for the time interval will look as 365*1/(0.95)^N, where N is the shopping act index. Its plot for the first 100 shopping trips is shown below.



Plot 2. The time interval (in hours) between shopping events as a function of shopping index

Let's now examine how the account value changes with time. For this, we need to calculate the total time elapsed from the start of the account for each shopping index N, which is a sum of all preceding time intervals, i.e. sum {0.365/(0.95)^(i)}, for all i=0, … ,N-1. The corresponding plot of the account value versus time elapsed is shown below.

One can clearly observe that the value-modulated discrete event behaviour of my wife lends itself to the characteristic that looks like a hyperbola. The analytic expression for this hyperbola can be derived using a geometric series sum, and is as follows: Value=1000*(1/(a*time +1)), where a=(1-K)/A, and K is a depreciation step (in our case 0.95) and A is the time scaling factor 0.365 which enables us to count time in days, hence Value=365/(0.05*time + 0.365).

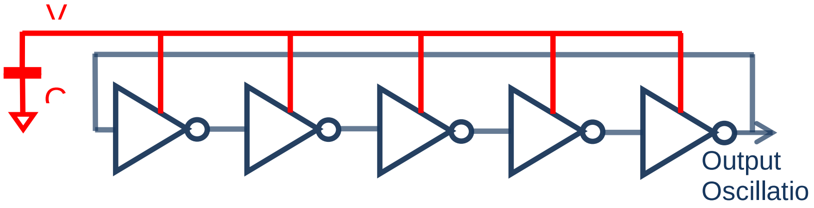


Plot 3. Account value vs time elapsed in days (after the first 100 shopping acts)

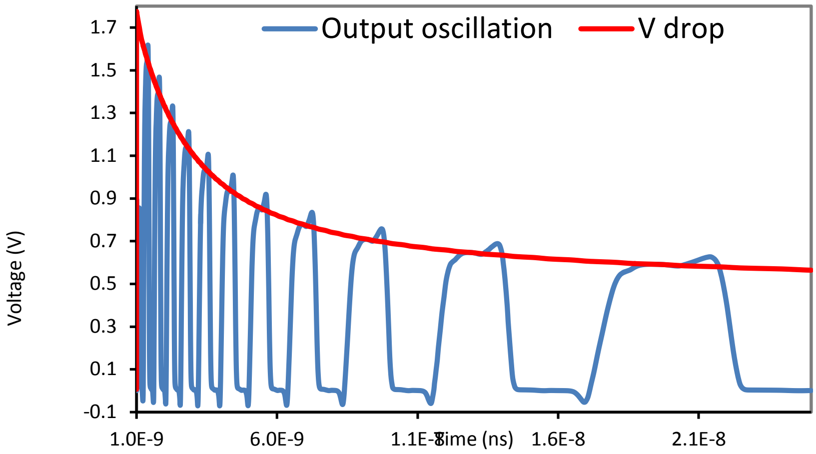
Anecdote 2: Capacitor discharging through a switching circuit

Now, one might ask, how does this anecdote relate to energy consumption? The answer is not obvious as yet, though some intuition has already been created in our above consideration of the fact that the "level of need to do a shopping" act was somehow related to the amount of money left on the account, which is akin to the power level applied to a system that performs discrete actions, such as for example computations.

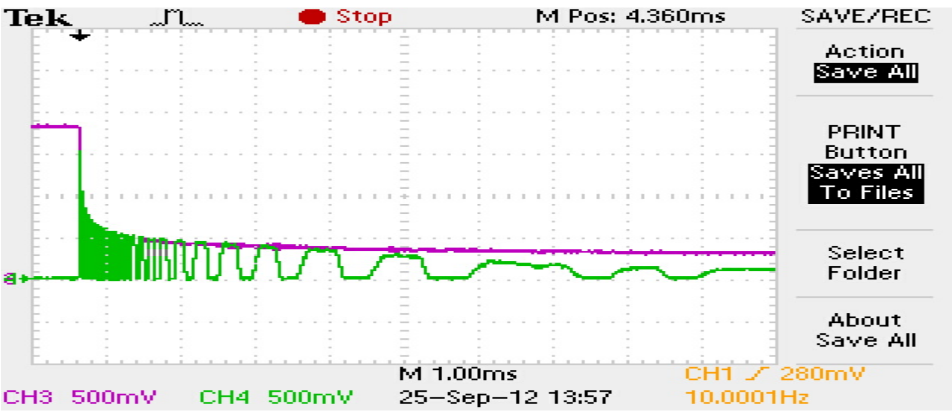
A few months ago Reza Ramezani, my PhD student, and I staged an experiment with an electronic circuit, consisting of a capacitor, a ring oscillator built from CMOS inverters and a switch. The capacitor was first charged to a certain initial value and then the switch would connect the capacitor to the power (so called Vdd) terminals of all the inverters in the oscillator. The plot of the capacitor's voltage drop against time is shown in the figure below. This curve is remarkably similar in its character to the above account depreciation curve. Together with Reza, we performed a more careful analysis of this relationship using a technique similar to our above "shopping index", and derived analytical expressions for two operational modes that the transistors in the oscillator can be subjected to, super-threshold and sub-threshold. Each mode defines the "time stretch" law, i.e. the way how the delay in the inverters of the oscillator changes with the current value of the supplied voltage. In the super-threshold region, for example, the delay is almost inversely proportional (with a specific power law) to the supply voltage. This leads to the voltage drop versus time to be hyperbolic. When I showed these results to my colleagues to my surprise they were quite surprised to see a curve that was neither exponential nor sinusoidal or a mix, but something of a power law type.



A capacitor and oscillator whose power is supplied by the capacitor



The hyperbolic process of capacitor discharging through the digital oscillator



The hyperbolic process in the voltage sensor based on charge-to-digital conversion

After a bit of thinking about the nature of the process a hyperbolic character of the curve seems quite natural for this sort of process, where the dynamics of the behaviour of the inverters in the circuit is completely determined by the power level provided by the capacitor, and the latter is obviously not constant.

The two papers whose references are listed below will take you to the details of this behaviour in the electronics world.

Meanwhile I am planning to produce another memo which will outline further ideas about hyperbolic processes, including interpreting hyperbolas as exponentials with ‘varying time constants’, composition of hyperbolic processes and other aspects laying foundations to the new type of hybrid (analogue-digital) computational structures.

Recommended reading:

1. R. Ramezani and A. Yakovlev, Capacitor discharging through asynchronous circuit switching, Proc. ASYNC'13, Santa Monica, CA, May 2013, IEEE CS Press, pp. 16-22, DOI 10.1109/ASYNC.2013.11

2. R. Ramezani, A. Yakovlev, F. Xia, J. Murphy and D. Shang, Voltage Sensing Using an Asynchronous Charge-to-Digital Converter for Energy-Autonomous Environments, IEEE Journal of Emerging and Selected Topics in Circuits and Systems (JETCAS), vol.3, No. 1, pp. 35-44, March 2013; doi: 10.1109/JETCAS.2013.2242776